Evaluation of Financial Instruments Possessing Non-Conventional Cash Flow

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Abstract
Investments are often justified and accepted based on the IRR as the main criterion of profitability. However, that criterion is hardly ever used to evaluate some financial instruments (e.g. short sales, options, futures and swaps). This is partially due to the fact that some instruments possess a cash flow describing a borrowing rather than an investment. Others have a non-conventional cash flow and, consequently, the IRR may be meaningless or impossible to determine. We describe a non-conventional cash flow of a financial instrument as a non-conventional project consisting of a sequence of single-period (simple) projects. Each simple project has only two cash flows with opposite signs therefore the IRR for the simple project is always determined. If there is a decomposition in which each simple project has the same IRR value, then that value is the IRR of the non-conventional project. If a decomposition of the non-conventional project into simple projects with the same IRR is impossible, the non-conventional project's IRR does not exist. If a simple project is an investment then the IRR is a rate of return for an investor. If a simple project is a loan then the IRR is an interest rate for the borrower, but not for the investor. Therefore the NPV method estimates a non-conventional project for two different participants simultaneously that leads to problems with definition of IRR. In order the loan's IRR would be a rate of return for the investor, but not an interest rate for the borrower, the sign of IRR should be replaced to opposite one. The paper discusses how to use the Generalized Net Present Value (GNPV) method to calculate a yield of the financial instrument with non-conventional cash flow. The function GNPV(r, p) depends on two rates: finance and reinvestment ones that determine a cost of funding and a rate of return, respectively. The equation GNPV (r, -r) = 0 is investigated in the paper. The solution of that equation is the Generalized Average Rate of Return (GARR). We suggest using the GARR as a new measure of a yield for evaluating financial instruments possessing a non-conventional cash flow and estimating a portfolio's performance over period with contributions and withdrawals.

Keywords: non-conventional cash flows, problems of the IRR, short sales, valuing swaps, rate of return of swap, portfolio performance evaluation, time weighted rate, money weighted rate, generalized net present value method

JEL-classification: G1, G 23, G31, O16, O22

Introduction

The concept of "return" is a basic theme in finance theory and finance textbooks [Alexander et al., 2001; Brealey et al., 2011; Markowits, 1952]. The terms "return" and "risk" determine the most fundamental principle of the investment area, namely, that greater risk requires higher return. As the concept of the time value of money (TVM) based on the discounted cash flow method (DCF) was developed, "return" was transformed into the internal rate of return (IRR), which became one of the key criteria for justifying and accepting investment. However, when the DCF theory is applied to analyze investment in stock market instruments, the results could be ambiguous. Moreover, according to Bos and Walker [2007], many authors fail to calculate the rate of return when analyzing short sales and derivatives (options, futures and swaps).

One of the indicators of portfolio management efficiency is the portfolio's rate of return in the holding period. The simplest approach to the determination of this indicator is calculating the weighted average rate of return for the entire portfolio in the holding period using the weighted average formula. However, the result of the calculation is only adequate for portfolios without withdrawals or contributions [Alexander et al., 2001; Fabozzi, 2002].

In order to eliminate this disadvantage, investors split the period of portfolio evaluation into sub-periods without withdrawals or contributions, and calculate the rate of return using the simplest approach (Return on Investment, ROI). A portfolio's rate of return throughout the period of evaluation is calculated as a weighted average rate of return for all the sub-periods using, as a rule, a time-weighted rate of return (TWR) that shows the effectiveness of every dollar invested during the evaluated period. The TWR does not consider portfolio size changes over the period. That method ignores contributions and withdrawals to and from the portfolio during the period over which the return is to be measured. The TWR is believed to reflect the manager's rather than the investor's performance as it is the investor but not the manager who takes the decisions concerning a portfolio's withdrawals and contributions.

The money-weighted rate of return (MWR) or the IRR can be applied along with the TWR. The MWR method takes into account all the contributions and withdrawals. It is used when you are trying to measure the performance experienced by an investor. Although the IRR is a better indicator, in theory, it is not as widespread in practice as the TWR because in case of major withdrawals and contributions comparable to the portfolio size the IRR may have several values or be indeterminable. This disadvantage of the IRR is well-known in the theory of investment analysis [Brealey et al., 2011; Brigham and Gapenski, 1996].

The multiple IRR and no IRR problems can arise when a project has non-conventional cash flows and the IRR is not a project's rate of return. Many scientists have tried to solve these problems [Athanasopoulos, 1978; Beaves, 1988; Bernhard, 1979; Hajdasinski, 1987; Hartman and Schafrick, 2004; Hazen, 2003; Lin, 1976; Magni, 2010; Mao, 1966; McDaniel et al., 1988; Rousse, 2008; Chiu and Escalante, 2012; Shull, 1992; Solomon, 1956; Teichroew et al., 1965]. The problem of determining the rate of return for non-conventional projects is related to the NPV rather than the IRR. It was noted that the task cannot be resolved within the bounds of the NPV method [Eschenbach and Nicholls, 2012; Kulakova and Kulakov, 2012]. Kulakov and Kulakova [2013] have recently proposed the GNPV method that extends and generalizes rather than replaces the NPV method in accordance with the continuity principle: every new theory has to be compatible with its predecessor, incorporating it as a limit case. The GNPV method relies on the solid theoretical background of the NPV approach. The NPV function depends on a single argument; it is, therefore, critical, even in the case of conventional projects, to determine whether the IRR is a rate of return on investment or an interest rate of borrowing? The GNPV method unequivocally uses two different rates for financing and reinvestment. The rollback method is used to compute the Present Value of project cash flows. The main advantage of that procedure is the possibility to obtain the project's present values at different periods. If the project's present value in a certain period is positive, the internal (finance) discount rate is used, otherwise, the external (reinvestment) rate is applied. The internal rate determines a cost of funding investment, whereas the external rate determines a return on investment.

The $GNPV(r, p)$ is a function of two variables. The GNPV roots can be sought as function $r = r(p)$ or $p = p(r)$ depending on the purpose of project evaluation. The Generalized Internal Rate of Return (GIRR) is a rate of return and represents the highest interest rate on the loan borrowed to finance the project (if all funds to finance are borrowed), with the resulting income of the current project used to repay the principal amount and the accrued interest. The $GIRR\ (p)$ is a function of the reinvestment rate. The Generalized External Rate of Return (GERR) is a rate of cost and represents the lowest rate of return on the external investment in which the borrowed funds can be invested to generate sufficient income to repay the loan with the accrued interest. The $GERR(r)$ is a function of the finance rate. The GIRR and GERR turn into the IRR in the case of conventional projects.

We use the GNPV method to determine a profitability of some financial instruments possessing non-conventional cash flows. But we cannot use that approach directly to calculate the rate of return of the portfolio with withdrawals and contributions. The GNPV method assumes one can choose to reinvest or not to reinvest the withdrawals. As a result, the portfolio performance will depend on how effectively you invest the funds outside the portfolio. The evaluation of the portfolio's profitability will be incorrect! To solve this problem we used the approach suggested by Bos and Walker [2007].

According to Bos and Walker, the problem of correct IRR determination based on the DCF method for short sales is related to cash flow formation: the negative cash flow follows the positive cash flow. In case of investments, we observe an inverted order, with the negative cash flow preceding the positive cash flow. The classical approach to the rate of return of short sales results in the IRR being negative in case of the profit and positive in case of the loss. To eliminate the resulting contradiction, Bos and Walker suggested using two rates of return: the "investment" Rate of Return (ROR) for long sales and the "borrowment" ROR for short sales. One of the authors' conclusions is that the rates cannot be added without taking into account the opposite sign of the "borrowment" ROR when calculating the rate of return of portfolios including both long and short sales. It is the same as "simply summing US dollars and Japanese yen without converting one of the currencies".

The problem of evaluating a loan has been solved in the theory of capital budgeting. Nevertheless, the difference between Bos and Walker's approach and the NPV method is that they estimate a loan project for an investor but not a borrower. They considered a portfolio with withdrawals and contributions over a single period. We consider a portfolio with withdrawals and contributions creating non-conventional cash flow over the holding period. We propose the Generalized Average Rate of Return (GARR) as a new approach without the disadvantages inherent to the IRR.

The paper proceeds as follows. Section II describes the GNPV method, justifies the application of equal rates with opposite signs in the function *GNPV*(*r*, -*r*) and introduces the concept of the GARR with its economic interpretation. Section III discusses how the GNPV method is used to calculate the rate of return of some transactions, an equity swap and a portfolio with withdrawals and contributions over the period. Section IV concludes.

The Generalized Average Rate of Return

Definition of the GNPV

The conventional project is always a net investment or a net borrowing [Bussey and Eschenbach, 1992; Hazen, 2003; Teichroew et al., 1965]. According to Bos and Walker, the projects containing two cash flows with different signs are classified as:

- "investment" if the initial cash flow is negative and the final cash flow is positive;
- "borrowment" if the initial cash flow is positive and the final cash flow is negative.

Depending on the type of project, the authors solve the equation *NPV* (*r*) = 0 and get two internal rates of return: the investment rate of return (IROR) and the borrowment rate of return (BROR). The authors do not consider "mixed" or non-conventional projects [Teichroew et al., 1965; Blaset Kastro and Kulakov, 2016] containing more than two cash flows and changing a sign more than once, because that would lead to the problem of the rate of return determination for non-conventional projects [Brealey et al., 2011; Brigham and Gapenski, 1996]. The problem of the IRR determination for non-conventional projects cannot be solved within the bounds of the NPV method because the NPV method uses a single discount rate. Kulakov and Kulakova [2013] have recently proposed the GNPV method that allows us to calculate the rate of return for non-conventional projects.

The GNPV function generalizes the NPV function by introducing two discount rates: the "internal" and the "external" ones or the "investment" and the "borrowment" respectively using Bos and Walker's terminology[1]. The GNPV function is determined by consistently discounting cash flows from the end to the beginning of the project using the rollback method. If the present value of the project in a certain period is positive, we use the internal discount rate, otherwise the external one. The internal rate determines a cost of funding an investment, and the external rate determines a return of return on an investment. The GNPV function is determined as follows:

$$PV_N = CF_N,$$

$$PV_i = \begin{cases} \dfrac{PV_{i+1}}{(1+r)} + CF_i, \ \text{if } PV_{i+1} > 0, \ \text{otherwise} \\ \dfrac{PV_{i+1}}{(1+p)} + CF_i, \ where\ i = N-1, \ \ldots, 0; \end{cases} \tag{1}$$

$$GNPV(r, p) = PV_0,$$

where CF_i is the project's cash flow at period *i*, ($i = N, \ldots, 0$); PV_i - the project's present value at period *i*; *r* and *p* are the internal and the external discount rate, respectively.

To determine the profitability of a non-conventional project we should solve the equation:

$$GNPV\,(r, p) = 0 \tag{2}$$

The solutions to equation (2) can be sought in the form of such functions as *r* = *r*(*p*) or *p* = *p*(*r*) depending on the purpose of non-conventional project evaluation [Kulakov and Kulakova 2013; Blaset Kastro and Kulakov 2017]. If we need to evaluate the project as an investment, it is necessary to solve the equation (2) with respect to the internal discount rate *r*. The solution is the function *r*(*p*) which determines the rate of return on the investment according to the conventional point of view. It represents the highest interest rate on the loan borrowed to finance the project (if all funds to finance are borrowed) so that the loan with accrued interest could be repaid by the income generated by the project. That rate of return called *Generalized Internal Rate of Return* (GIRR) is similar to the IRR in case of conventional investment projects.

[1] In the Capital budgeting theory the "internal" and "external" rates are known as finance and reinvestment rates respectively.

If the non-conventional project is considered as a loan, i.e. a source to finance another project, the equation (2) needs to be solved with respect to the external discount rate *p*. The solution is the function *p*(*r*) which determines the loan interest rate (rate of cost). That rate is called the *Generalized External Rate of Return* (GERR). It represents the lowest rate of return on the external investment in which the borrowed funds can be invested so that the debt with accrued interest could be repaid. The GERR is the same as the IRR in case of conventional borrowing projects.

Bos and Walker [2007] noted that the rate of return and the return have the same sign in case of investment projects, while the corresponding signs of the borrowing projects are opposite. This is accounted for by the fact that the IRR of borrowing projects is the borrower's but not the investor's rate of return. In order to estimate a borrowing project from the investor's point of view, one needs to change the sign of the external rate.

Thus, we have considered the function *GNPV*(*r*) = *GNPV*(*r*, -*r*).

The GNPV as the base of GARR

To simplify the study of the GNPV properties, we apply the continuously compounded rate ρ called "force of interest" instead of the discrete rate *r*. The force of interest reflects the increase of an accumulated sum within an infinitely short period of time. The relation between the rate for the period and the continuously compounded rate is as follows:

$$(1+r)=e^{\rho},\quad \frac{1}{(1+r)}=e^{-\rho},$$

where $-1<r<1;\ -\infty<\rho<\infty$.

The following recurrent formula is used to determine the present value of cash flows at time *i* in terms of the continuously compounded rate:

$$PV_i = PV_{i+1}e^{-\rho}+CF_i.$$

Theorem. The function *GNPV* (ρ, –ρ) monotonically decreases as the discount rate ρ increases.

Proof.

Consider the *GNPV* (ρ) determined as follows:

$$PV_N = CF_N,$$

$$PV_i = \begin{cases} PV_{i+1}e^{-\rho}+CF_i, & \text{if } PV_{i+1}>0, \text{ otherwise} \\ PV_{i+1}e^{\rho}+CF_i, & \text{where } \rho>0,\ i=N\text{-}1,\ \dots,\ 0; \end{cases} \qquad (3)$$

$$GNPV(\rho)=PV_0,$$

The derivative of the present value at time *i* with respect to rate ρ is equal to:

$$\frac{dPV_N}{d\rho}=\frac{dCF_N}{d\rho}=0,\quad i=N,$$

$$\frac{dPV_i}{d\rho}=\frac{d}{d\rho}(PV_{i+1}e^{-\rho})=(-PV_{i+1}+\frac{dPV_{i+1}}{d\rho})e^{-\rho},\quad i=N\text{-}1,\ \dots,\ 0.$$

If $\forall\ PV_{i+1}>0$ then the derivative $\frac{dPV_i}{d\rho}<0$

and, consequently, the function $PV_i(\rho)$ monotonically decreases as the rate ρ increases. Let us assume that for $\forall$ *i* = *N*, … , *k*+1; $PV_i>0$ and $PV_k=PV_{k+1}e^{-\rho}+CF_k<0$ then $PV_{k-1}=PV_ke^{\rho}+CF_{k-1}$.

Let us calculate the derivative of the present value $\frac{dPV_{k-1}}{d\rho}$ at time (*k*–1):

$$\frac{dPV_{k-1}}{d\rho}=\frac{d}{d\rho}(PV_ke^{\rho}+CF_{k-1})=\frac{dPV_k}{d\rho}e^{\rho}+PV_ke^{\rho}=(PV_k+\frac{dPV_k}{d\rho})e^{\rho}.$$

The derivative

$\frac{dPV_k}{d\rho} < 0$ and the present value $PV_k < 0$,

then $\frac{dPV_{k-1}}{d\rho} < 0$.

Consequently, the present value $PV_k(\rho)$ at time (k–1) is a monotonically decreasing function of the discount rate ρ. Continuing the calculation until time $t = 0$,

we get $\frac{dGNPV(\rho)}{d\rho} < 0$.

Thus, the *GNPV*(ρ) monotonically decreases as the discount rate increases.

Here, it follows that the equation *GNPV*(*r*, -*r*) = 0 can have at most one real root!

Economic significance

The corollary to the theorem is that the function *GNPV*(*r*) has at most one root for -1 < *r* < 1. We called the rate bringing the *GNPV*(*r*) down to zero the *Generalized Average Rate of Return* (GARR).

$$GNPV\,(GARR) = 0. \qquad (4)$$

The GARR can be interpreted as an investor's constant rate of return for pure investment and pure borrowing subprojects comprising a non-conventional project. This constant rate is "average" according to the definition because it is the same for all the subprojects.

To better understand the essence of the GARR and its difference from the IRR we will use the method of one-period rates proposed by Hazen [2003]. Hazen has shown the IRR to be "a constant one-period rate *k* for a cash flow stream ***x*** if and only if there exists an investment stream ***c*** which yields ***x*** at the constant per-period rate of return *k*". We will replace the investment stream with cash flow. Any project may be presented as a sequence of one-period projects. Let us divide the *i*-th cash flow into two parts: $CF_i = a_i + b_i$. Suppose the *i*-th one-period project consists of two cash flows: b_i and a_{i+1}, which are the initial and the final cash flows respectively. The initial and the final cash flows are given as follows:

$$\begin{aligned} & a_0 = 0, \quad b_0 = CF_0, \\ & a_i = -\begin{cases} b_{i-1}(1+r), \text{ if } b_{i-1} < 0 \\ b_{i-1}(1+p), \text{ if } b_{i-1} > 0 \end{cases} \quad \begin{matrix} b_i = CF_i - a_i, \\ \text{where} \quad i = 1,\dots,N, \end{matrix} \\ & a_N = CF_N, \end{aligned} \qquad (5)$$

where *r* and *p* are the finance and reinvestment rates respectively.

The system of equations (5) is another way of writing the equations (1, 2). Let us confirm this statement.

$$a_N = CF_N,\ b_{N-1} = -\frac{a_N}{(1+r)} = -\frac{CF_N}{(1+r)} = CF_{N-1} - a_{N-1} \Rightarrow a_{N-1} = \frac{CF_N}{(1+r)} + CF_{N-1} = PV_{N-1}.$$

Let $a_i = PV_i$ then $b_{i-1} = -\frac{PV_i}{1+r} = CF_{i-1} - a_{i-1} \Rightarrow a_{i-1} = \frac{PV_i}{1+r} + CF_{i-1} = PV_{i-1}$.

The constraint $a_0 = 0$ is equivalent to *GNPV*(*r*, *p*) = 0.

We also have: $b_i = CF_i - a_i = b_{i-1}(1+r) + CF_i \Rightarrow FV_{i-1}(1+r) + CF_i = FV_i$, where FV_i is the future value of cash flows [Teichroew et al., 1965]. Thus the initial cash flow of one-period project *i* is the project's future value at period *i*: $b_i = FV_i$. The final cash flow of the *i*-th one-period project is the project's present value at the period (*i*+1): $a_i = PV_{i+1}$. One might say two rates (*r*, *p*) satisfying the system of equations (5) form a sequence of one-period projects. The finance rate links cash flows of investments, and the reinvestment rate links cash flows of borrowings.

When the equation *GNPV*(*r*, *r*) = 0 has real solutions, then the rate *r* is the IRR, i.e. it is the rate of return of one-period investments for an investor and the interest rate of one-period loans for a borrower. Therefore the IRR can be the rate of return of the whole non-conventional project if all the loans could be really reinvested at the IRR. However that reinvestment assumption is not always right and can lead to two problems: the equation *GNPV*(*r*, *r*) = 0 could have any real solutions or no real solutions at all. The equation *GNPV*(*r*, -*r*) = 0 has at most one real root according to the Theorem. In this case, the root *r* is the GARR, i.e. it is a constant rate of return for all one-period investments and borrowings estimated for an investor. Therefore the GARR could be the rate of return of the whole non-conventional project estimated for an investor.

Discussion

Let us consider some financial instruments and evaluate its using the GNPV method. To give a better understanding of the GNPV method at first we will evaluate the financial instruments containing only two cash flows with opposite signs (single-period or simple project).

Long sales

Let us consider two cases of long sales:

1) An investor buys stocks for \$100 at time t=0 and sells them for \$110 at time t=1 getting a \$10 return over the period. As the final cash flow is positive, it must be discounted using the "internal" discount rate r. According to (1) the GNPV is equal to:

$$GNPV = CF_0 + \frac{CF_1}{(1+r)},$$

where CF_0 is the initial cash flow at time t=0, CF_1 is the final cash flow at time t=1. At $r = GIRR$ the GNPV is equal to zero. Thus:

$$GNPV = CF_0 + \frac{CF_1}{(1+GIRR)} = \\ = 0 \Rightarrow GIRR = -\frac{CF_0 + CF_1}{CF_0}. \qquad (6)$$

Substituting cash flow values (CF_0 = –100, CF_1 = +110) in equation (6), we get the rate of return of the long sale:

$$GIRR = -\frac{-100+110}{-100} = -\frac{10}{-100} = +10\%.$$

2) An investor buys stocks for \$100 at time t=0 and sells them for \$90 at the end of the period (time t =1). Thus, the initial cash flow CF_0 = –100 and the final cash flow CF_1 = +90. Substituting these values in equation (6), we get the rate of return of the long sale:

$$GIRR = -\frac{-100+90}{-100} = -\frac{-10}{-100} = -10\%.$$

The investor gets a \$10 return in the first case and a \$10 loss in the second case of the long sales. The GIRR is the investor's rate of return in both cases, i.e. 10% and -10% respectively. When assessing a long sale the GNPV method turns into the NPV method, and GIRR coincides with the IRR.

Short sales

Let us consider two cases of short sales.

1) An investor short sells stocks for \$100 at time t=0 and redeems them for \$110 at time t=1 with a - \$10 loss. This case is similar to taking out a loan: an investor (as a borrower) obtains a \$100 loan at t=0 and repays the amount of \$110 at t=1, taking into account \$10 interest for using the loan. Thus, the cash flows are equal to +100 and -110, respectively. As the final cash flow is negative, it must be discounted using the "external" discount rate p. The application of the GNPV method gives:

$$GNPV = CF_0 + \frac{CF_1}{(1+p)} \Rightarrow CF_0 + \frac{CF_1}{(1+GERR)} = \\ = 0 \Rightarrow GERR = -\frac{CF_0 + CF_1}{CF_0} = \frac{110-100}{100} = 10\%.$$

The GNPV method estimates the transaction as the loan for the borrower but not the investor. The GERR is the loan interest rate and not the rate of return. It is a contradiction from the point of view of the investor: the investor received a \$10 loss, but the rate of return of the transaction equals to 10%. To evaluate the transaction from the point of view of the investor one should use the external rate with the opposite sign, i.e. $p = -r$.

$$GNPV = CF_0 + \frac{CF_1}{(1-r)} \Rightarrow CF_0 + \frac{CF_1}{(1-GARR)} = 0,$$

whence it follows that

$$GARR = \frac{CF_0 + CF_1}{CF_0}$$

The investor's rate of return of the short sale in this case is:

$$GARR = \frac{100-110}{100} = \frac{-10}{100} = -10\%.$$

Consequently, the GARR of the short sale is negative if the investor sustains a loss, i.e. the GARR has the same sign as the result of the transaction.

In financial textbooks, the rate of return of the short sale is calculated differently [Alexander et al., 2001]. The investor does not receive the money when he short sells a stock without cover; he puts up a margin considered as an investment. Let the margin requirement be 60% of the stock price, then (CF_0 = – \$60). When the investor buys a stock back and also gets the margin back (CF_1 = \$100 – \$110 + \$60 = \$50). Thus, the rate of return on a short sale is: (\$50/\$60 – 1 = –16.7%). For the sake of better understanding, we will not take into consideration the margin requirements or the interest earned on short proceeds, commissions or other transaction fees. As the composition of short sales in a portfolio is less than the net asset value, the margin requirements are supported. Therefore, we will only consider cash flows of sales and purchases.

2) An investor short sells stocks for \$100 at time t=0 and redeems them for \$90 at time t=1 getting a \$10 return. The investor's rate of return on a short sale is:

$$\boxed{GARR = \frac{100-90}{100} = \frac{10}{100} = 10\%.}$$

The GARR's sign is the same as the return sign. Consequently, the GARR of the short sale is positive if the investor gets a profit. Thus the GARR is the rate of return of both investments and borrowings for the investor.

Table 1: Cash flows of one-period transactions and two-period sequence

Period	0	1	2	Return	IRR	Yield
Short sale	$40	-$25		$15	-37.5%	37.5%
Long sale		-$95	$90	-$5	-5.3%	-5.3%
Sequence of S-L sales	$40	-$120	$90	$10	50%	?

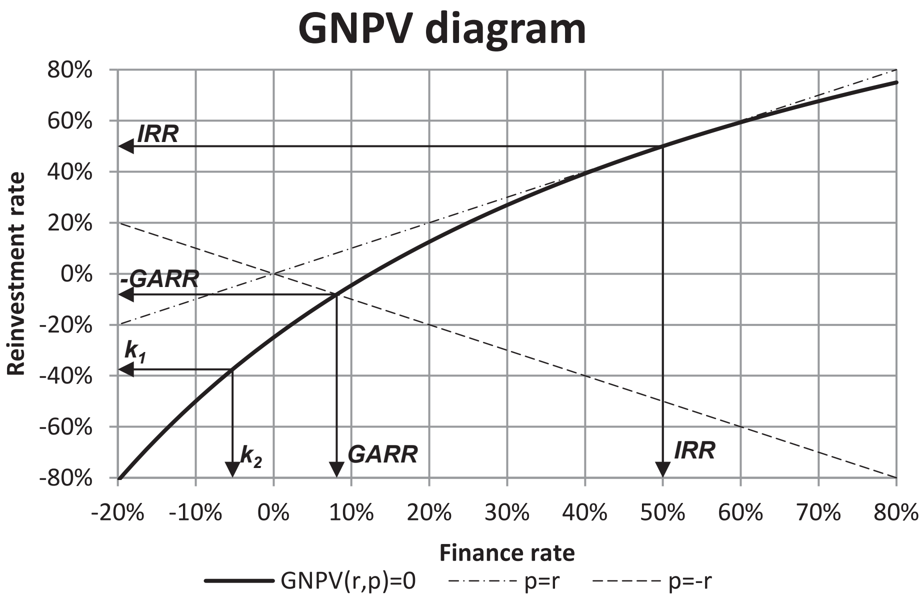
Table 2: The cash flows of short and long sales corresponding to the tangency point

Period	0	1	2	Return	IRR	Yield
Short sale	$40	-$60		-$20	50%	-50%
Long sale		-$60	$90	$30	50%	50%
Combination of S-L sales	$40	-$120	$90	$10	50%	?

Table 3: The cash flows of short and long sales corresponding to the intersection point

Period	0	1	2	Return	IRR	Yield
Short sale	$40	-$36.8		$3.2	-8.1%	8.1%
Long sale		-$83.2	$90	$6.8	8.1%	8.1%
Combination of S-L sales	$40	-$120	$90	$10	50%	8.1%

Figure 1: The IRR and GARR determination by using the GNPV diagram.



The sequence of short and long sales

Let us now consider a transaction consisting of a sequence of short and long sales. First the investor short sells stocks for $40 at time t=0. Thereafter he redeems the stocks for $25 and buys other stocks for $95 at time t=1. Finally, the investor sells stocks for $90 at time t=2 getting a $10 return over two periods. Table 1 shows the cash flows of one-period transactions and two-period sequence. The rate of return (yield) of the short sale is 37.5% [($40-$25)/$40], the yield of the long sale is -5.3% [($90-$95)/$95]. The IRR of the transaction sequence is 50%.

The cash flow of the transaction sequence is non-conventional, so the IRR can be meaningless. To explain that statement let's consider the GNPV diagram (Figure 1) plotted for the short and long sale sequence. The solid line consists of a set of points with coordinates (r, p) at which the function $GNPV(r, p)$ equals zero. In the case of two-period project the rates (r, p) might be determined as one-period return rates (k_1, k_2) [Hazen, 2003; Magni, 2010]. The rates k_1 and k_2 correspond to the IRRs of the two one-period transactions from Table 1 (k_1 = -37.5%, k_2 = -5.3%). There is an infinite set of such pairs of rates. Each pair is determined by equation: Price of Short purchase + Price of Long purchase = $120. For example, the pair (50%, 50%) is given by $60+$60=$120. The cash flows of those two transactions are shown in Table 2. Note: the yield and the IRR of the first transaction (short sale) have opposite signs. This result follows the NPV method since the short sale is a borrowing and its IRR is defined for a borrower. To calculate the average rate of return of a sequence of investments and borrowings we should transform the loan interest rates into the rates of return.

Let us consider two points on curve $GNPV(r, p) = 0$. The first point (IRR, IRR) is the point of tangency of the line $p(r) = r$ with the curve $GNPV(r, p) = 0$. The cash flows of short and long sales corresponding to the tangency point (IRR, IRR) are presented in Table 2. We can see that the yield of the short sale is equal to -50% and the yield of the long sale equals 50%. Therefore the IRR cannot be the yield of a combination. The second important point (GARR, -GARR) is the intersection point of the line $p(r) = - r$ and the curve $GNPV(r, p) = 0$. The cash flows of the short and long sales corresponding to the point (GARR, -GARR) are presented in Table 3. We can see that the yields of the short sale and long sale are equal to each other and equal the yield of the sequence of short and long sales (GARR = 8.1%).

We can also estimate the average yield of two transactions as the ratio of the total inflows sum to total outflows sum. This ratio is equal to 8.3% [($40 + $90) / ($25 + $95)] and almost coincides with the GARR.

Thus, the GARR is the constant or average rate of return for all one-period projects comprising the non-conventional project.

Equity swaps

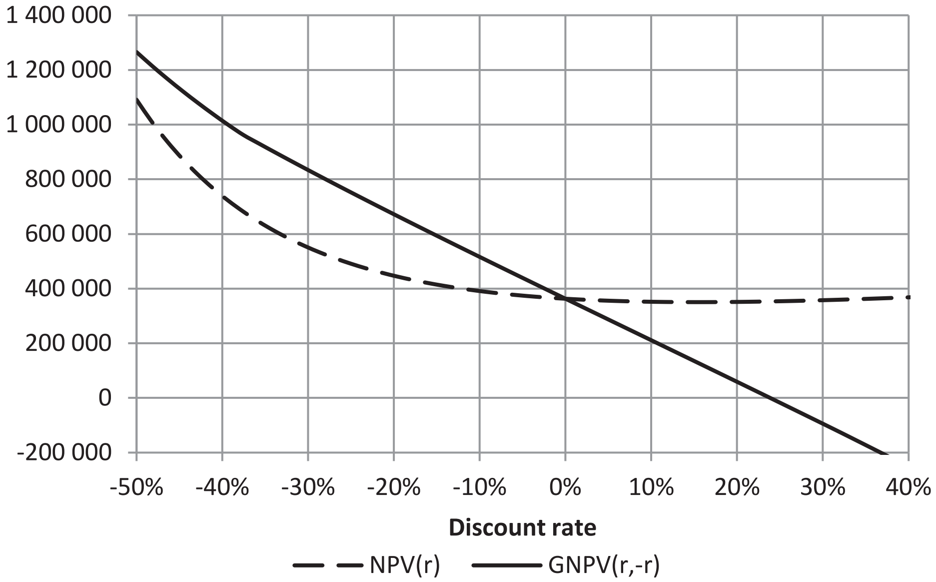
Let us consider an equity swap: two parties make a series of payments to each other with at least one set of payments determined by a stock or index return. The other set of payments can be a fixed or floating rate or the return on another stock or index [Alexander et al., 2001]. These series of payments occur on regularly scheduled dates over a specified period of time. An example of an equity swap is shown in Table 4. On December 15 of a given year, the first party enters into a swap to pay a fixed rate of 10% with payment terms of 90/360 and receive the return on the S&P 500 with payments to occur on March 15, June 15, September 15, and December 15 for two years. Payments are calculated on a notional principal of $20 million (Table 4).

Table 4: Hypothetical Payments on Two-Year Equity Swap with Quarterly Settlement to Pay a Fixed Rate of 10% and Receive the Return on the S&P 500 on Notional Principal of $20 Million.

Date	S&P 500 Index	Return on S&P 500	S&P 500 Cash Flow	Fixed Interest Cash Flow	Net Cash Flow
December 15	1 176.0				
March 15	1 295.0	10.1190%	2 023 810	500 000	1 523 810
June 15	1 319.0	1.8533%	370 656	500 000	-129 344
September 15	1 316.0	-0.2274%	-45 489	500 000	-545 489
December 15	1 200.0	-8.8146%	-1 762 918	500 000	-2 262 918
March 15	1 289.0	7.4167%	1 483 333	500 000	983 333
June 15	1 369.0	6.2064%	1 241 272	500 000	741 272
September 15	1 353.0	-1.1687%	-233 747	500 000	-733 747
December 15	1 440.0	6.4302%	1 286 031	500 000	786 031

It appears to be impossible to calculate the rate of return of cash flows for each swap's party using conventional techniques. Fig. 2 shows the NPV function of swap's cash flows (the dashed line) depending on the discount rate r. The NPV function does not cross the X-axis, therefore it has no real roots and the internal rate of return of a swap cannot be determined. The other average rates of return are meaningless. Unlike the $NPV(r)$ function, the $GNPV(r)$ decreases monotonically as the discount rate (the solid line) increases and is zero at $r = 23.9\%$. This value is the rate of return for the first party, whereas the rate of return for the second party has the opposite sign and equals -23.9%.

Figure 2: The NPV and GNPV depending on the discount rate.



Portfolio Performance Evaluation

As a rule, the following two indicators are used to measure the rate of return of a portfolio: the time-weighted rate of return (TWR) and the money-weighted rate of return (MWR) or the internal rate of return [Fabozzi, 2002; Bodie et al., 2005]. The first approach ignores the number of stocks held over the period. The second approach takes into account contributions to and withdrawals from the portfolio made over the period. The TWR measures the results attributable to the investment manager. The MWR reflects both the performance of the manager and the timing of investor transactions. The study of the advantages and disadvantages of the above approaches is beyond the scope of this paper, bearing in mind that other approaches such as the arithmetic, geometric and weighted average rates of return analyze other aspects of portfolio performance. Let us examine a hypothetical portfolio (Table 5) using the above-mentioned rates of return.

Table 5: Hypothetical portfolio

Date	Cash Flow	Portfolio Value	Quarterly Return (%)
01.01.2012	7 000	7 000	
31.03.2012		10 000	42.9%
01.04.2012	-8 000	2 000	
01.07.2012		2 600	30.0%
30.09.2012		3 000	15.4%
01.10.2012	9 000	12 000	
01.01.2013		9 000	-25.0%

The annual rates of return of this portfolio are presented in Table 6.

Table 6: The annual rates of return of the hypothetical portfolio

Rate of Return	Value per year (%)
Arithmetic average	63.2
Geometric average/TWR	60.7
MWR/IRR	30.9
GARR	26.3

Thus, the difference between the widely applied TWR and MWR is almost two-fold. This can be accounted for by the fact that the size of the portfolio in the two middle quarters of the year was four times less than in the first and fourth quarters. Therefore, the impact of higher returns in the middle of the year is less in the MWR calculation and the TWR ignores the returns altogether. The real rate of return of the portfolio, i.e. the GARR, is less than the MWR/IRR because the MWR/IRR approach implicitly assumes that the free funds of the second period have been reinvested at the IRR that could not have been the case.

Conclusion

The IRR is a rate of return of an investment project only with conventional cash flows. In case of a project with non-conventional cash flows the IRR is not a rate of return and cannot exist at all. Some financial instruments can have non-conventional cash flows, for example: a combination of short and long sales, an interest rate swap, a portfolio over holding period with withdrawals and contributions. Therefore the IRR is not always applicable to measure the rate of return of such instruments.

In this work the index GARR is presented as a measure of the rate of return. That index is deduced from the GNPV method that has been recently proposed by Kulakova and Kulakov [2012, 2013]. The GNPV function is determined by two discount rates (finance and reinvestment) unlike the NPV function which is determined by a single rate. These rates reflect the rate of cost and the rate of return. Due to the use of the two rates, the financing and reinvestment processes of cash flows are separated. Every project with non-conventional cash flow can be estimated from two points of view: as an investment and as a loan. The investment rate of return is a function of the reinvestment rate, and the loan interest rate (rate of cost) is a function of the finance rate.

The conventional project is always either an investment or a loan. The NPV method uses only one rate, therefore the IRR is either the rate of return of an investment or the interest rate of a loan. To estimate a loan for the investor and not the borrower, it is necessary to use the interest rate with opposite sign. To evaluate an investment for the borrower the sign of the finance rate should be change to opposite. We suggested to use the substitution $p = -r$ and investigated the function $GNPV(r) = GNPV(r,-r)$. We proved that the function $GNPV(r)$ decreases monotonically as the discount rate r increases in the range $-1 < r < 1$ and therefore can have only one root in that range. If the root exists it is the rate of return of the non-conventional project considered as a sequence of the one-period projects for the investor. The GARR is a constant or average rate of return for all single-period projects comprising the non-conventional project.

We used the GNPV method to calculate a profitability of some financial instruments possessing non-conventional cash flows (a combination of short and long sales, an interest rate swap) when the index IRR is unreasonable or does not exist at all. We also showed that in calculating a portfolio performance over holding period with withdrawals and contributions the IRR can give an overestimated value as compared to the GARR. That is because the IRR method assumes the money withdrawn from the portfolio is invested at the IRR while the reinvestment does not actually take place.

As a new measure the Generalized Average Rate of Return could be used for evaluating financial instruments having non-conventional cash flows and estimating the portfolio's performance over period with contributions and withdrawals.

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