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Typical Uncertainty of Sustainable Growth of ESG Enterprises in a Dynamically Effective Stochastic Economy with Public Debt

Vasiliy Belykh
Candidate of Physico-Mathematical Sciences, Associate Professor at the Department of Economics, Corporate Management and Finance, Novosibirsk State University of Economics and Management, Novosibirsk, Russia, v.v.belykh@gmail.com, ORCID

Abstract

The subject of the study is an enterprise based on ESG principles, which implies its concern for the interests of not only the current but also future generations. The study was undertaken to develop a sustainable growth model that takes into account economic uncertainty and is capable of covering a period of time sufficient for ESG program implementation. Despite the widespread desire to achieve sustainable growth, there is no unanimity in regard to the methods of integrating long-term development programs with traditional commitment to current profit maximization. One of the difficulties is associated with reaching intergenerational planning horizons. In addition to their short-term nature, existing sustainable growth models do not include an indicator of the uncertainty associated with the activities of the enterprise. To solve this problem, the paper uses methods of integrating stochastic differential equations, which allows to move away from the deterministic dependencies of predecessor models. The resulting stochastic trend model takes into account the systematic long-term impact of the environment. This approach turns it into a hyper-long-term planning tool commensurate with the duration of the enterprise life cycle. According to this model, the probability density of revenue growth rate is subject to a logarithmically normal law with numerical characteristics changing under the influence of competition and inflation. The paper envisages several development scenarios characterized by different dynamics of the random component. If the enterprise follows ESG principles, then the typical growth scenario will be the most suitable one. The random component of growth of a typical enterprise degenerates over time, and its rate is determined by the risk-free interest rate. From the concept of intersecting generations, it follows that typical enterprises contribute to the dynamic efficiency of the economy.

Keywords: stochastic process, volatility trend, growth rate, financial ratios, hyperlong-term, dynamic efficiency

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Introduction

To ensure sustainable economic development, one has to manage uncertainty that influences key business decisions: amount of investment, expense structure, pricing policy, planning horizon. Modern enterprises attempt to reduce it in various ways: diversification, long-term agreements, subscription for services, hedging of price and exchange rate risks, and also implement sustainable development programs based on ESG principles. The distinctive feature of hedging is its hyper-long-term horizon, which encompasses several generations [1]. Empirical data about the influence of ESG on enterprises' financial performance is ambiguous. A review of the practice of implementation of ESG transformation programs demonstrates a positive effect in 60% of cases, neutral – in 9%, mixed – in 25%, and negative – in 6% [2]. These figures suggest that some companies are unable to implement ESG without damaging the core value measured by profit [3]. It is difficult to transition to the sustainable development track due to the underdevelopment of hyper-long-term planning methods. The existing studies are premised on probability-based models, which can confirm statistical relationships between the analyzed indicators but are unsuitable for operational business planning.

The situation with ESG implementation may be rectified by means of a sustainable development trend model that allows to combine hyper-long-term financial objectives with current indicators [4]. The model describes the enterprise's revenue dynamics over a time interval comparable with its life cycle span. The addition of long-term trends of financial ratios to the model allowed to shift to this modelling horizon. One of possible trend combinations describes how the enterprise moves towards the equilibrium state. However, the trend model fails to take into consideration the economic uncertainty factor, thus, it becomes less feasible. The present research eliminates this gap by adding the standard deviation of revenue to the list of financial ratios. Since the existing empirical data about dynamics of its uncertainty is insufficient, we had to use economic theory when describing this indicator. In order to determine the uncertainty level that corresponds to the enterprise's equilibrium state, we used a stochastic model of intersecting generations with safe assets (public debt) and dynamic efficiency [5]. Exponentiality of uncertainty is introduced similarly to that of other financial ratios of a typical enterprise whose dynamics have been thoroughly studied empirically.

Probability density of financial variables' growth rate often comprises heavy tails, which account for the excessive frequency of extreme events. Since the implementation of ESG programs decreases the enterprises' dependence on rare but strong shocks [6], it seems acceptable to use a lognormal distribution that does not take into account abnormalities of a frequency distribution to describe the revenue growth rate. The existing empirical data proves that this assumption is true in four cases out of seven [7]. It is comparable to the statistics of successful ESG implementation. In spite of the unchanged standard deviation of the logarithm of the revenue growth rate over a long period of time for the majority of enterprises, in a considerable number of cases it tends to decrease gradually. No satisfactory explanation of this volatility phenomenon has been offered yet by corporate finance theory. Along with long-term changes, the influence of short-term external shocks such as economic crises, manufacturing innovations, a sudden change in prices and demand for products, on the uncertainty level has been understudied. Such factors may cause structural changes in dynamic properties of enterprises' time series and require further research.

The purpose of the study is to develop a stochastic trend model of sustainable development capable of covering the whole enterprise's life cycle. To achieve this goal, revenue is described by a stochastic process where drift and volatility are influenced by competition and inflation. The model is intended for practical application in ESG transition planning.

Theory

Due to economic uncertainty, revenue S_t at time t has a random value. To model this variable, we apply geometric Brownian motion whose cross-section complies with the logarithmically normal law:

$$dS_t = g_t S_t dt + \sigma_t S_t dW_t , \quad (1)$$

where g_t and σ_t are the drift and volatility parameters; W_t – Wiener process ($t \geq 0$). Parameters g_t and σ_t in equation (1) are time-dependent because the enterprise growth rate and the uncertainty level related to its operations change along with its life cycle stages. Usually, an enterprise demonstrates the fastest growth at the initial stages, and subsequently its development slows down. The sustainable development stage is the longest, and the last one with a positive growth rate. At this stage, programs aimed at decreasing the uncertainty level are implemented .

Drift

The drift parameter accounts for the rapidity of average revenue growth rate. In order to account for the influence of the external environment, we use the sustainable development trend model [4], where the rapidity of revenue growth rate – taking into account the changes caused by inflation and competition – is described by the following equation:

$$g_t = p_0 + \sum_{k=1}^{27} p_k e^{q_k t} , \quad (2)$$

with constants equal to:

$$p_0 = h_0 f_0 c_0 (1+l_0) + h_0 f_0 y_0 (1-\phi)(1+l_0) + h_0 n_0 \varepsilon (1+l_0) - n_0 \alpha (1-\phi) h_0 l_0 ,$$

$$p_1 = \left[h_1 f_0 c_0 + h_1 f_0 y_0 (1-\phi) + h_1 n_0 \varepsilon \right](1+l_0) - n_0 \alpha (1-\phi) h_1 l_0 , \qquad q_1 = \delta,$$

$$p_2 = h_0 f_0 c_1 (1+l_0) , \quad q_2 = \gamma, \qquad p_3 = h_0 f_0 y_1 (1-\phi)(1+l_0) , \quad q_3 = \eta,$$

$$p_4 = h_0 n_1 \left[\varepsilon (1+l_0) - \alpha (1-\phi) l_0 \right], \quad q_4 = \pi, \qquad p_5 = h_0 f_1 (1+l_0)\left[c_0 + y_0 (1-\phi) \right], \quad q_5 = \xi,$$

$p_6 = h_0 l_1 \left[f_0 c_0 + f_0 y_0 (1-\phi) + n_0 \varepsilon - n_0 \alpha (1-\phi) \right]$, $q_6 = \lambda$,

$p_7 = h_1 f_0 c_1 (1+l_0)$, $q_7 = \gamma + \delta$, $p_8 = h_0 f_0 c_1 l_1$, $q_8 = \gamma + \lambda$,

$p_9 = h_1 l_1 \left[f_0 c_0 + f_0 y_0 (1-\phi) + n_0 \varepsilon - n_0 \alpha (1-\phi) \right]$, $q_9 = \lambda + \delta$,

$p_{10} = h_0 f_0 y_1 (1-\phi) l_1$, $q_{10} = \eta + \lambda$, $p_{11} = h_0 n_1 l_1 \varepsilon - n_1 \alpha (1-\phi) h_0 l_1$, $q_{11} = \pi + \lambda$,

$p_{12} = h_0 f_1 c_1 (1+l_0)$, $q_{12} = \gamma + \xi$, $p_{13} = h_1 f_1 (1+l_0) \left[c_0 + y_0 (1-\phi) \right]$, $q_{13} = \xi + \delta$, (3)

$p_{14} = h_0 f_1 y_1 (1-\phi)(1+l_0)$, $q_{14} = \eta + \xi$, $p_{15} = h_0 f_1 l_1 \left[c_0 + y_0 (1-\phi) \right]$, $q_{15} = \xi + \lambda$,

$p_{16} = h_1 f_0 y_1 (1-\phi)(1+l_0)$, $q_{16} = \eta + \delta$, $p_{17} = h_1 n_1 \varepsilon (1+l_0) - n_1 \alpha (1-\phi) h_1 l_0$, $q_{17} = \pi + \delta$,

$p_{18} = h_1 f_0 c_1 l_1$, $q_{18} = \gamma + \lambda + \delta$, $p_{19} = h_1 f_1 c_1 (1+l_0)$, $q_{19} = \gamma + \xi + \delta$,

$p_{20} = h_1 f_1 l_1 \left[c_0 + y_0 (1-\phi) \right]$, $q_{20} = \xi + \lambda + \delta$, $p_{21} = h_0 f_1 y_1 (1-\phi) l_1$, $q_{21} = \eta + \xi + \lambda$,

$p_{22} = h_0 f_1 c_1 l_1$, $q_{22} = \gamma + \xi + \lambda$, $p_{23} = h_1 n_1 \varepsilon l_1 - n_1 \alpha (1-\phi) h_1 l_1$, $q_{23} = \pi + \lambda + \delta$,

$p_{24} = h_1 f_1 y_1 (1-\phi)(1+l_0)$, $q_{24} = \eta + \xi + \delta$, $p_{25} = h_1 f_0 y_1 (1-\phi) l_1$, $q_{25} = \eta + \lambda + \delta$,

$p_{26} = h_1 f_1 c_1 l_1$, $q_{26} = \gamma + \xi + \lambda + \delta$, $p_{27} = h_1 f_1 y_1 (1-\phi) l_1$, $q_{27} = \eta + \xi + \lambda + \delta$,

where ϕ – profit tax rate; ε – the share of assets subject to additional evaluation due to price inflation, which are financed from equity and debt capital; α – the share of debt with free floating interest rate.

Equation (2) was obtained by introducing trends of financial ratios into the sustainable development inflation model [8; 9], including: investment b, asset turnover (S/A), return on sales m, financial leverage (D/E), the spread between the inflation rates of product and manufacturing resource prices z and the rate of inflation of manufacturing resource prices j, which reveal a change in statistical expectations of ratios over time:

$$E_t(b) = h_0 + h_1 e^{\delta t}, \qquad E_t\left(\frac{S}{A}\right) = f_0 + f_1 e^{\xi t},$$

$$E_t(m) = c_0 + c_1 e^{\gamma t}, \qquad E_t\left(\frac{D}{E}\right) = l_0 + l_1 e^{\lambda t}, \qquad (4)$$

$$E_t(z) = y_0 + y_1 e^{\eta t}, \qquad E_t(j) = n_0 + n_1 e^{\pi t},$$

where $h_0, f_0, c_0, l_0, y_0, n_0$ – final values responsible for extreme values of financial ratios when $t \to \infty$; $h_1, f_1, c_1, l_1, y_1, n_1$ – deviations from final values at time $t = 0$; $\delta, \xi, \gamma, \lambda, \eta, \pi$ – rates of deviation increment.

Since models [8; 9] provided the basis for equation (2), here, as in those models:

economy-wide inflation determines changes in manufacturing resource prices of an enterprise and the nominal interest rate on borrowed funds, and is added to the real rate according to the Fisher effect;

the investment ratio takes into consideration the amount of invested funds, as well as additional capital related to the increases in the value of an enterprise's assets, which is a result of an inflation-related increase in their nominal value.

Volatility

The volatility parameter controls the random change in the analyzed indicator. We use an exponential function to describe its time dependence:

$$\sigma_t = \sigma_0 e^{q_{28} t}, \qquad (5)$$

where q_{28} – the rapidity of revenue volatility growth rate. According to (5), the direction of the volatility trend depends on the sign of q_{28}. If this constant is below zero, uncertainty decreases over time, if the constant equals zero – it remains constant, if the constant takes on a positive value – it increases. In practice, the rapidity of volatility growth rate may be calculated by comparing the volatility of several parts of a time series of the revenue growth rate logarithm.

Model

By applying (2) and (5) in (1), we obtain a stochastic differential equation of revenue:

$$dS_t = \left(p_0 + \sum_{k=1}^{27} p_k e^{q_k t} \right) S_t dt + \sigma_0 e^{q_{28} t} S_t dW_t. \qquad (6)$$

After integrating it (see Appendix 1), we have the following stochastic model of the revenue growth rate logarithm for time points t and $t+T$:

$$\ln\left(\frac{S_{t+T}}{S_t}\right) = p_0 T + \sum_{k=1}^{27} \frac{p_k}{q_k} \left[e^{q_k (t+T)} - e^{q_k t} \right] - \frac{\sigma_{T,t}^2}{2} + \int_t^{t+T} \sigma_0 e^{q_{28} \theta} dW_\theta, \qquad (7)$$

where $\sigma_{T,t} = \sigma\left[\ln\left(\frac{S_{t+T}}{S_t}\right) \right]$ – standard deviation of the revenue growth rate, which equals:

$$\sigma_{T,t} = \sigma_0 \left(\frac{e^{2q_{28}(t+T)} - e^{2q_{28}t}}{2q_{28}} \right)^{0,5}. \quad (8)$$

Since financial ratios' trends allow to account for changes in corporate operating performance on a long-term horizon, model (7) covers the entire life cycle of the enterprise. The final values of trends produce the most significant influence in the revenue growth trajectory. It may be typical, logistic, exponential or ending in the decline stage [4]. If the growth rate of financial ratios, including volatility, equals zero, then after integrating (6), we obtain geometric Brownian motion that describes growth in an environment without external influence.

Data

The primary information source was the SKRIN database. We selected companies with non-zero reporting for 1998–2018 out of the enterprises with the largest revenue as of 1998. Subsequent analysis was differentiated by industry according to the Russian National Classifier of Types of Economic Activity (OKVED) (Table 1). Using the criterion offered in paper [10], we checked time series for structural changes. In spite of the 2008 economic crisis, market fluctuations, and technological innovations, no structural changes were detected in the overwhelming majority of enterprises. To analyze the dynamics of uncertainty, we verified statistical hypotheses about the variances of the first σ_1^2 and second σ_2^2 halves of the time series of the revenue growth rate logarithm and evaluated the volatility growth rate q_{28}. The reason for choosing this method was the limited span of the enterprises' time series, which renders the use of more complex calculations statistically unjustified. Besides, it should be noted that this method is of limited accuracy given the expected nonlinearity of the analyzed indicator. Variances were compared at the significance level of $\alpha = 0.05$. The speed q_{28} was calculated using equation (8), assuming that variances σ_1^2 and σ_2^2 are in the middle of the halves of the time series. The distribution density of the volatility growth rate is demonstrated using enterprises engaged in the extraction of commercial minerals as an example (Figure 1). The presented data reveals that at the industry level, the tendency towards decreasing volatility prevails, while at the enterprise level, the model with constant volatility is more relevant. There may be a discrepancy present because the analyzed period is not sufficiently long, and q_{28} is too low for a statistically significant change in uncertainty to be found for all enterprises. The macroeconomic uncertainty index also indicates a slow rate of uncertainty decline (see Appendix 2). Thus, empirical analysis fails to provide an unambiguous answer concerning the direction of revenue volatility over a long-term horizon. Consequently, the issue of its exponential decrease at the enterprise level remains unsolved.

Statistical Modelling

We are going to consider the results of statistical modelling of the revenue growth rate logarithm using equation (7) (Figure 2). The dashed and solid lines show the results of calculations when $\sigma_0 = 0$. The enterprise's drift parameter is calculated on the basis of the average values of the financial ratios for JSC Kuzbasskaya Toplivnaya Company for 2001–2021. Random trajectories were constructed using truly random numbers from random.org. The rapidity of the volatility growth rate $q_{28} = -0.044$ corresponds to the average value for enterprises engaged in extraction of commercial minerals. Over time, the logarithm of the revenue growth rate gravitates towards to the level of p_0 (see Figure 2), which is indicative of the marginal growth rate. A decrease in the uncertainty level causes a degeneration of the stochastic component.

Table 1. Inconstancy of the uncertainty level

Industry	Sample size	Presence of structural changes	Number of enterprises with a confirmed hypothesis		Mean value q_{28}
			$\sigma_1^2 > \sigma_2^2$	$\sigma_1^2 < \sigma_2^2$	
Extraction of commercial minerals	68	5	13	0	-0.044
Commercial manufacturing	356	5	34	1	-0.037
Supply of electric power, energy, gas and steam; air conditioning	32	2	12	1	-0.065
Wholesale and retail; vehicle and motorbike repair	107	4	17	1	-0.056
Transportation and storage	58	12	9	0	-0.057
For all enterprises	621(100%)	28(4.5%)	85(14%)	3(0.5%)	-0.045

Figure 1. Density of the distribution of the revenue volatility growth rate

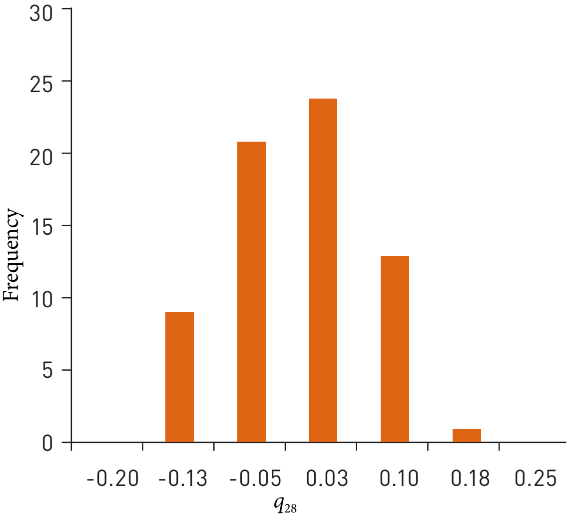
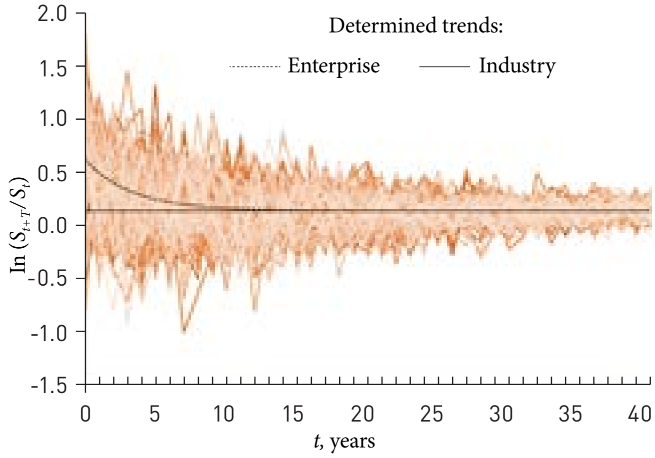


Figure 2. Logarithm of the revenue growth rate



Discussion

Hyper-long-term Nature of ESG Transformation

The ESG approach to operating a business has become the standard for enterprises seeking sustainable development [11]. To achieve actual changes in the environmental, social, and governance (ESG) spheres, it is necessary to apply ESG principles for a long time and go beyond strategic planning since there is a delayed effect of investment in sustainable growth. Model (7) meets these requirements and allows to model growth at a time horizon comparable with the length of the enterprise's life cycle. At the same time, it models sustainable growth, which is the key goal of ESG transformation.

Typical Enterprise and Proactive Planning

With different final trend values (4), the growth trajectory describes natural, typical, logistic growth or ends in a period of decline. Given that ESG programs are focused on sustainable development, the most suitable conditions for their implementation are offered in a typical development scenario aimed at achieving a stable growth stage. The financial indicators of a typical enterprise tend towards the levels established by industry medians [12–14]. The emergence of clear financial benchmarks limit the management's attempts to over-indulge in current profits at the expense of long-term development. There emerges an opportunity for proactive planning [15, p. 84]. It allows the enterprise not merely to respond to changes but to predict the trajectory of its development in advance, thus creating an ESG agenda and the environment for its implementation.

From Typical Uncertainty to Dynamic Efficiency

Model (7) allows for decreasing, sideways and growing trends of uncertainty. If revenue uncertainty is typical, then, similarly to other financial ratios of a typical enterprise, it demonstrates a negative rapidity of growth rate. Thereby, we obtain the dependence with $t \to \infty$ instead of (7):

$$S_t = S_0 e^{p_0 t}, \qquad (9)$$

where p_0 accounts for the rapidity of the revenue growth rate at the deterministic growth stage. As long as there is no uncertainty, p_0 corresponds to risk-free return. The rapidity of an enterprise's equity growth rate takes on the same value because its financial ratios remain unchanged. According to the intersecting generations model, if the capital growth rate and the risk-free return are equal, it is possible to avoid overaccumulation of capital, achieving the dynamic efficiency of stochastic economy with public debt [5]. At the enterprise level, this corresponds to a typical growth scenario, which allows to achieve a balance between current profit and long-term ESG objectives.

Factors of Uncertainty Decrease

ESG programs are not among "quick" volatility reduction tools, but as the effects from reducing regulatory, reputational and operational risks accumulate, they gradually reduce volatility:

- implementation of environment protection technologies (E) guards against regulatory and price risks. Revenue fluctuations related to a decline in production caused by accidents and administrative prohibitions decrease. Transition to renewable power sources guards against the volatility of fossil fuel prices;
- responsible social policy (S) drives consumer loyalty. Such enterprises have more stable demand even during financial crises. The risks of conflicts with local communities, strikes and consumer boycotts that may bring down sales disappear;
- improvement of corporate governance (G) mitigates corruption risks and reduces the potential for financial fraud, which causes reputational damage. This increases the number of reliable contract partners who maintain continuous renewal of production facilities and stable sales.

Conclusion

In spite of a general consensus concerning the prospects of the ESG economy, the issues of enterprises' transition to the sustainable development track have been understudied. The existing studies focus on the analysis of the consequences caused by introducing economic incentives and disciplinary measures that encourage ESG implementation, while the issues related to adapting corporate planning to new requirements remain unsolved. Such a one-sided approach makes it difficult to practically implement ESG programs. Hyper-long-term planning tools, including the stochastic trend model of sustainable growth, point towards a solution. Empirical data is indicative of two volatility trends that correspond to the constant and gradually decreasing levels of revenue uncertainty. The final conclusion about the prevailing direction of changes may be made after a study of time series exceeding twenty years. The offered model supports both versions and indicates that there is a connection between the latter one and the typical growth scenario. When this scenario is implemented, revenue growth rate gravitates towards the risk-free rate, thus fostering the economy's movement towards dynamic efficiency. When the latter is achieved, there is no need in non-market mechanisms for stimulating ESG programs.

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Appendix 1

Integration of a Stochastic Differential Equation of Sustainable Growth

We are going to solve the stochastic differential equation for steadily growing revenue:

$$dS_t = \left(p_0 + \sum_{k=1}^{27} p_k e^{q_k t}\right) S_t dt + \sigma_0 e^{q_{28} t} S_t dW_t, \quad (10)$$

where S_t – stochastic process; p_0, p_k, q_k, σ_0 and q_{28} – constants;
t – time $(t \quad 0)$; W_t – Wiener process.

We use the Ito formula for functions $F(S_t, t) = \ln S_t$, so that:

$$dF(S_t, t) = \varphi(t) dt + \sigma_0 e^{q_{28} t} dW_t, \quad (11)$$

where

$$\varphi_t = \frac{\partial F(S_t, t)}{\partial t} + \left(p_0 + \sum_{k=1}^{27} p_k e^{q_k t}\right) S_t \frac{\partial F(S_t, t)}{\partial S_t} + \frac{\left(\sigma_0 e^{q_{28} t} S_t\right)^2}{2} \frac{\partial^2 F(S_t, t)}{\partial S_t^{\,2}} = p_0 + \sum_{k=1}^{27} p_k e^{q_k t} - 0{,}5\sigma_0^2 e^{2 q_{28} t}. \quad (12)$$

The solution of the stochastic differential equation (11) is described by the following process:

$$F(S_t, t) = F(S_0, 0) + \int_0^t \varphi(\theta) d\theta + \int_0^t \sigma_0 e^{q_{28}\theta} dW_\theta, \quad (13)$$

where S_0 – value of the stochastic process at the initial time point.

We get the first integral on the right side of the equation (13):

$$\int_0^t \varphi(\theta) d\theta = \int_0^t p_0 d\theta + \int_0^t p_1 e^{q_1\theta} d\theta + \ldots + \int_0^t p_{27} e^{q_{27}\theta} d\theta - \int_0^t 0{,}5\sigma_0^2 e^{2q_{28}\theta} d\theta =$$
$$= p_0 t + \frac{p_1}{q_1}\left(e^{q_1 t} - 1\right) + \ldots + \frac{p_{27}}{q_{27}}\left(e^{q_{27} t} - 1\right) - \frac{\sigma_0^2}{4q_{28}}\left(e^{2q_{28} t} - 1\right). \quad (14)$$

Since $F(S_t, t) = \ln S_t$ according to (13) we have:

$$\ln\left(\frac{S_t}{S_0}\right) = p_0 t + \sum_{k=1}^{27} \frac{p_k}{q_k}\left(e^{q_k t} - 1\right) - \frac{\sigma_0^2}{4q_{28}}\left(e^{2q_{28} t} - 1\right) + \int_0^t \sigma_0 e^{q_{28}\theta} dW_\theta. \quad (15)$$

With $q_{28} \neq 0$ the standard deviation of process (15) equals

$$\sigma_t = \sigma\left[\ln\left(\frac{S_t}{S_0}\right)\right] = \left\{\int_0^t \left[\sigma_0 e^{q_{28}\theta}\right]^2 d\theta\right\}^{0,5} = \left[\frac{\sigma_0^2}{2q_{28}}\left(e^{2q_{28} t} - 1\right)\right]^{0,5}. \quad (16)$$

For revenue from (15) we obtain:

$$S_t = S_0 \exp\left[p_0 t + \sum_{k=1}^{27} \frac{p_k}{q_k}\left(e^{q_k t} - 1\right) - \frac{\sigma_0^2}{4q_{28}}\left(e^{2q_{28} t} - 1\right) + \int_0^t \sigma_0 e^{q_{28}\theta} dW_\theta\right]. \quad (17)$$

Random quantity (17) is log-normally distributed with statistical expectation equal to:

$$E(S_t) = S_0 \exp\left\{E\left[\ln\left(\frac{S_t}{S_0}\right)\right] + 0{,}5\sigma_t^2\right\} = S_0 \exp\left\{p_0 t + \sum_{k=1}^{27} \frac{p_k}{q_k}\left(e^{q_k t} - 1\right) - \frac{\sigma_0^2}{4q_{28}}\left(e^{2q_{28} t} - 1\right) + \frac{\sigma_0^2}{4q_{28}}\left(e^{2q_{28} t} - 1\right)\right\} =$$
$$= S_0 \exp\left[p_0 t - \sum_{k=1}^{27} \frac{p_k}{q_k}\left(1 - e^{q_k t}\right)\right]. \quad (18)$$

If $q_{28} = 0$ the standard deviation is described by the following formula:

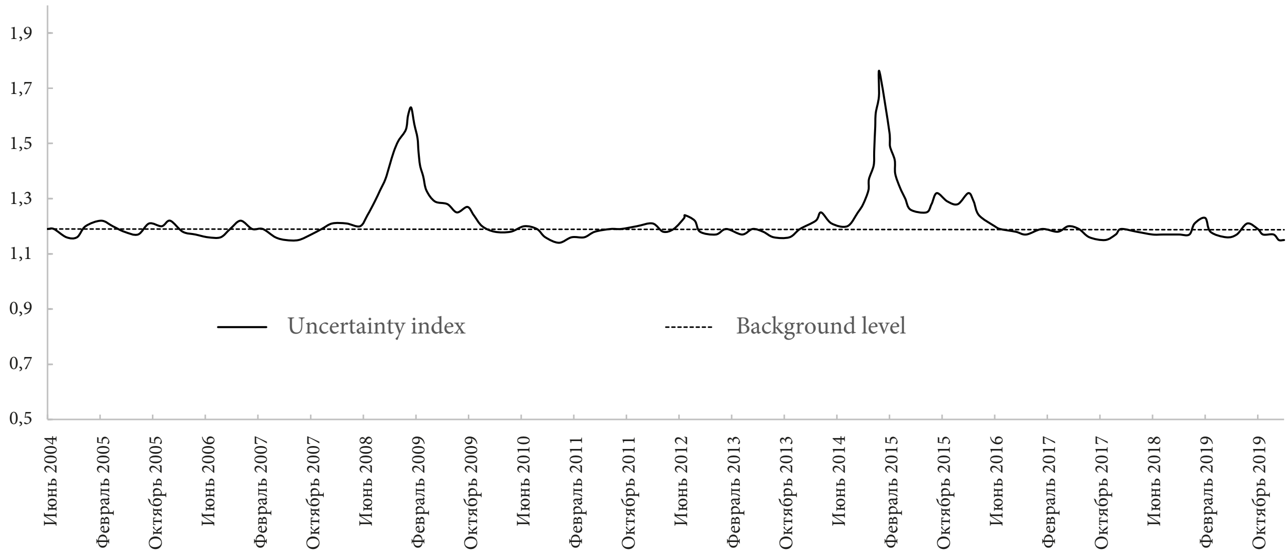
$$\sigma_t = \sigma\left[\ln\left(\frac{S_t}{S_0}\right)\right] = \left\{\int_0^t \sigma_0^2 d\theta\right\}^{0,5} = \left[\sigma_0^2 t\right]^{0,5}. \quad (19)$$

Appendix 2

Uncertainty Index

The macroeconomic uncertainty index [16] equal to the weighted average of standard deviations of forecast errors over a 12-month horizon calculated on the basis of 39 macroeconomic indicators of the Russian economy has a two-component structure (Figure 3). The barely noticeable monotonic decrease in the indicator is interrupted by the outbursts that mark the periods of macroeconomic shocks. After approximating the background level with an exponential equation, the rapidity of the annual growth rate of the index turned out to be –0.00015.

Figure 3. Macroeconomic Uncertainty Index



Source: [16].

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