CAPM-Like Model And the Special Form of the Utility Function

Dranev Yury[7]

The variance and semivariance are traditional measures of asset returns volatility since Markowitz proposed the market portfolio theory. Well known models for expected asset returns were developed under assumptions of mean-variance or mean-semivariance investor's behavior. But numerous papers provided arguments against these models because of unrealistic assumptions and controversial empiric evidence. More complicated models with downside risk measures experienced difficulties with applications. The new model based on the special form of the investor's utility function is proposed in this paper.

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Introduction

One of the basic assumptions of CAPM is the mean-variance behavior of investors. In other words investors seek to maximize the expected utility of their wealth by considering the first two moments of returns distribution i.e. the expected return and the variance. Substantial number of research papers questioned the adequacy of this assumption. In the mean-variance framework the variance can be an appropriate risk measure only in case of symmetrical distribution of returns. The symmetrical shape of the return's distribution is not supported by empirical evidence, especially in developing markets. Another drawback of variance in this context is the fact that it assigns equal weights to ups and downs of returns. This is not exactly the case since investors often prefer upside over downside volatility.

Asymmetric investors behavior forced researchers to seek another risk measure and utility function. Markowitz (1952) assumed that semivariance could be more plausible measure than the variance. Later on, Hogan and Warren (1974) and Estrada (2002, 2007) introduced models which were analogous to CAPM. They suggested that under so-called mean-semivariance behavior the appropriate measure of risk would be downside beta that can be calculated based on the cosemivariance of the asset and the market returns. More general model involving lower partial moments (LPM) was introduced by Bawa (1975) and Bawa and Lindenberg (1977). The main idea behind this model was Taylor expansion of investor utility function in the case of downside risk aversion.

Downside CAPM (or DCAPM) in the Estrada's version is rather straightforward for practical use but it has several substantial drawbacks. One of them is the fact that in the mean-semivariance framework investor excludes the upside movement from his risk consideration completely.

Several more general models were invented for the purpose of approximation of the utility function with higher moments (skewness and kurtosis). Researchers also studied logarithmic utility function to obtain better explanation of investor's behavior. However the empirical tests of these models provided mixed results compared to the CAPM (see for example Teplova Shutova (2011)).

This work attempts to apply the well-known concept of entropy to explain the investor behavior and the risk measure of an asset in the financial market. Later, the introduced risk measure is applied to develop the equation similar to that of CAPM for expected return.

[7] Senior Lecturer, Corporate finance department, NRU HSE

The entropic covariance of returns

The concept of entropy comes from thermodynamics and information theory. But recently entropy was extensively used for solution of financial mathematics problems (see, for example, Cherny, Maslov (2004)). Besides that, the so called cross-entropy (or Kullback–Leibler distance) was actively applied to portfolio optimization through fuzzy cross-entropy minimization procedure (Kapur (1992), Qin, Li, Ji (2009)).

Instead of applying minimal relative entropy measure for returns' distributions or the use of minimal cross-entropy of weights in portfolio optimization I will use the concept of entropy in a different way. Let $E(R_X)$ be the expectation of a random return R_X. Consider random market returns of two assets: R_X and R_Y. Assume that an asset price stays above zero and $R_X>-1$, $R_Y>-1$ (no short selling). Let D_X and D_Y be normed differences between asset returns and their means i.e. $D_X = \frac{R_X-E(R_X)}{1+E(R_X)}$ and $D_Y=\frac{R_Y-E(R_Y)}{1+E(R_Y)}$.

We can see that $1+D_Y=\frac{1+R_Y}{1+E(R_Y)}>0$ since no short selling is allowed.

Similar to covariance *the entropic covariance* between asset returns will be given by:

$$ECOV(R_X, R_Y)= (1+E(R_X)) (1+E(R_Y)) E (D_X Log(1+D_Y)) \quad (1)$$

First of all one may notice that $ECOV$ defined by (1) is not symmetric, i.e.

$$ECOV(R_X, R_Y)\neq ECOV(R_Y, R_X).$$

The next equation expresses relationship between the entropic covariance and the traditional covariance.

$$COV(R_X I(R_X \geq E(R_X)), R_Y)=E(R_X - E(R_X)(R_Y- E(R_Y) I(R_X \geq E(R_X)))=$$
$$(1+E(R_Y)) E(R_X - E(R_X))D_Y I(R_X \geq E(R_X)))\geq$$
$$(1+E(R_X)) (1+E(R_Y))E(D_X Log (1+D_Y) I(R_X \geq E(R_X)))= ECOV(R_X I(R_X \geq E(R_X)), R_Y) \quad (2)$$

Inequality $a>log(1+a)$ is used to obtain (2). Covariance of R_X and R_Y is bigger than the entropic covariance for values of R_X above the mean. For values of R_X below the mean the opposite is true. Assuming that variation of R_Y is small (or close to zero), i.e. R_Y is close to $E(R_Y)$, one may conclude that $COV(R_X, R_Y)$ is close to $ECOV(R_X,R_Y)$ and if variation of asset R_X is close to zero, then $COV(R_X, R_Y)$ is close to $ECOV(R_Y,R_X)$.

Taking a closer look at the $ECOV$ definition the reader may notice that (1) reminds of the form of Kullback–Leibler distance except for distributions which are replaced with differences between returns and mean returns.

It is necessary to mention another property of entropic covariance. Since

$$E(R_X - E(R_X))(Log(1+E(R_Y))-E Log(1+ R_Y))=0,$$

I conclude that

$$ECOV(R_X, R_Y)= (1+E(R_Y)) COV(R_X, Log(1+ R_Y)) \quad (3)$$

Hence entropic covariance can be deduced from covariance by changing variables.

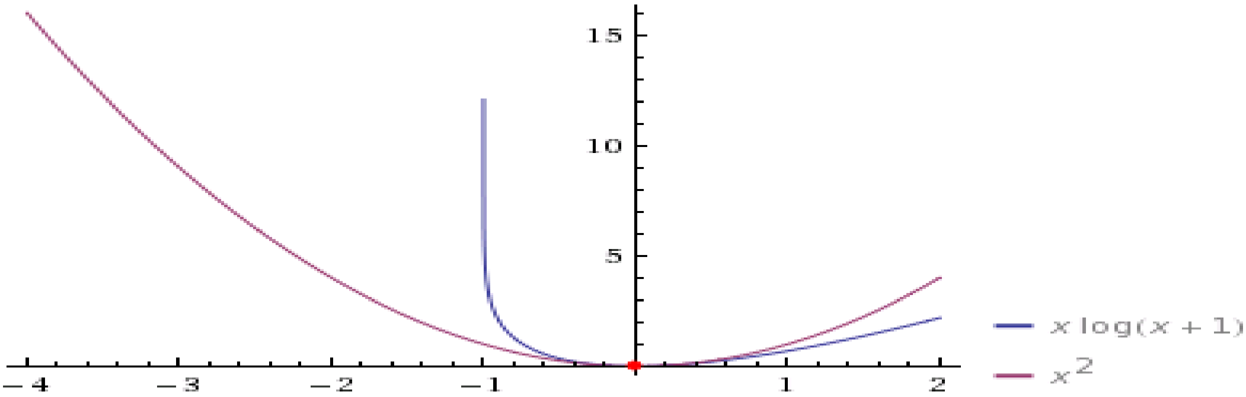
Consider now the entropic covariance of the return R_X with itself:

$$ECOV(R_X, R_X)= (1+E(R_X))^2 E (D_X Log(1+D_X)). \quad (4)$$

Since $a Log (1+a)$ is a convex function of a and $E(D_X)=0$ we can apply Jensen inequality to obtain:

$$E(D_X Log(1+D_X))\geq E(D_X)Log(1+E(D_X))= 0.$$

Hence the entropic variance defined by $EV(R_X)= ECOV(R_X, R_X)$ is positive, less than the variance for R_X greater than the mean and greater than variance for R_X less than the mean. Figure below illustrates the difference between the variance (quadratic distance to the mean) and the entropic variance (entropic distance to the mean of the form $x\ log(1+x)$).



Asymmetric behavior of the entropic variance is crucial for our purposes. Indeed, the entropic variance may be considered as a risk measure of asset returns that provides downside risk with the greater weight.

The beta coefficient and the CAPM based on the entropic covariance

Now we will simply follow Estrada (2007). First of all, mean-variance behavior of investor in the classical model can be substituted by mean – entropic variance behavior. From this point variables that describe the investor portfolio are indexed by *p*, variables that describe the behavior of the total market portfolio are indexed with *M* and parameters of the i-th asset and a risk-free asset are indexed by *i* and *f* respectively. The alternative investor utility function will be $U=U(\mu_p, EV_p)$ where μ_p is the mean of returns and EV_p is the entropic variance of investor's portfolio returns. The entropic standard deviation will be $ESD_p=\sqrt{EV_p}$.

We can now introduce the entropic covariance with respect to the benchmark portfolio return *B*. For this purpose I denote the normed distance between the asset return and the benchmark return by $D_X^B=\frac{R_X - B}{1+B}$. Then the entropic covariance with respect to the benchmark *B* is given by

$$ECOV\ (B,R_X,\ R_Y)=\ (1+B)^2\ E\ (D_X^B\ Log(1+D_Y^B)). \quad (5)$$

I am not discussing the efficient frontier and the capital market line for the mean entropic variance framework in details here. Avoiding relatively complicated mathematical computations I also omit the solution of portfolio optimization problem i.e. minimization of EV_p subject to certain conditions. For details and solution of a similar problem in LPM framework I would like to refer to Bawa (1977) and Harlow (1989). The only conclusion to be mentioned here is the fact that the capital market line is not linear in this case.

In order to draw an analogue of the security market line I am ready to introduce the entropic beta. Let R_M be the market return and R_i be the return of i-th asset. Then the *entropic beta* of an asset is given by

$$\beta_i^E = ECOV(R_i,\ R_M)/\ EV(R_M) = \frac{COV(R_i, Log(1+R_M))}{COV(R_M, Log(1+R_M))} \quad (6)$$

Such choice of beta is not determined by equation of slopes of curves that correspond to the market portfolio with the riskless asset and the market portfolio with i-th asset as in the classical framework (or as in Harlow (1989) for example). Beta as in (6) is proposed following the logic of replacing the input of the market portfolio to investor's portfolio i.e. R_M is partially replaced with Log(1+ R_M). By the way, R_M was not completely replaced by $Log(1+ R_M)$ in the denominator of (6) to make β_i^E dimensionless. Following (2) if $R_i<E(R_i)$ the asset *i* contributes more to the risk of portfolio than in case of traditional mean-variance framework. And if $R_i>E(R_i)$ it contributes less.

Another reasonable choice of beta in the entropic variance framework could be given by

$$\beta_i^E(R_f) = ECOV(R_f,\ R_i,\ R_M)/\ EV(R_f, R_M) \quad (7)$$

Equation (7) looks more complicated for practical use, that is why I prefer beta in the form of (6). To support my choice of entropic beta I should notice that mean returns as a benchmark works better for LPM framework as it was shown in Harlow (1989). Estrada (2007) also prefers mean returns in definition of downside beta.

Now the capital asset pricing model in entropic variance framework would be

$$E(R_i)=R_f+\beta_i^E\,(E(R_M)-R_f). \qquad (8)$$

Conclusion

The new model describing the expected return of assets was proposed in this paper. The considered model may demonstrate better explanation power than CAPM in the mean-variance framework and DCAPM in the mean-semivariance framework. This suggestion is based on assumption that investor does pay more attention to downside volatility but his attitude to upside volatility should not be completely excluded from consideration. The expected return of the asset given by (8) takes into account both downside and upside volatility of market returns with more weight attributed to downside movement.

The next step would be conducting empirical tests of (8). Empirical evidence for developed and emerging markets may show that the mean entropic variance framework is more plausible in some cases than other frameworks. Results of this research are going to be completed soon.

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